Exercise 27

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x\to 0} |x| = 0$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x - 0| < \delta \qquad \text{then} \qquad ||x| - 0| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x|$.

$$||x| - 0| < \varepsilon$$

$$||x|| < \varepsilon$$

$$|x| < \varepsilon$$

Choose $\delta = \varepsilon$. Now, assuming that $|x| < \delta$,

$$\begin{aligned} ||x| - 0| &= ||x|| \\ &= |x| \\ &< \delta \\ &= \varepsilon. \end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 0} |x| = 0.$$